Journal of Applied Mechanics and Technical Physics, Vol. 40, No. 1, 1999

ON THE EXISTENCE OF THREE SOLUTIONS FOR A SUPERCRITICAL STEADY FLOW OF A HEAVY FLUID OVER OBSTRUCTIONS

K. E. Afanas'ev and S. V. Stukolov

UDC 532.5

The paper is devoted to the solution of the steady problem of ideal incompressible fluid flow over a semi-circular cylinder located at the bottom. Calculations showed that the problem has at least three solutions for the Froude number. In the absence of an obstruction at the bottom, the proposed algorithm allows one to construct solitary waves up to limiting waves. The paper reports the most important wave characteristics: circulation, mass, and potential and kinetic energy. Analysis of the calculation results leads to the conclusion that all maximum values of the solitary-wave characteristics are attained before the maximum amplitude and the maximum of the mass does not coincide with the maxima of the total energy and the Froude number.

Heavy fluid flows have been the subject of extensive research. For example, Kiselev and Kotlyar [1] solved the problem of supercritical flow of a heavy fluid in a channel with a curvilinear bottom and drew the important conclusion that when the shape of an obstruction at the bottom is symmetric, the shape of the free surface is also symmetric. However, for flow over a mound at Froude numbers (Fr) close to unity, the problem has a nonunique solution. Moiseev [2] was the first to prove this fact. The problem of constructing two solutions in an exact nonlinear formulation was studied numerically by Guzevskii [3], who showed that in the absence of an obstruction, the first solution corresponds to a rectilinear flow, and the second describes a solitary wave. Instead of the Froude number, Guzevskii [3] introduced the parameter $V = v_0/v_\infty$, which describes the ratio of the velocity v_0 of the wave crest to the free stream velocity v_∞. Thus, the Froude number is a function of V: $\mathrm{Fr} = \mathrm{Fr}(V)$. The introduction of the parameter V ensures a unique solution of the problem of flow over obstructions and allows one to construct waves (up to limiting waves) over the entire range of Froude numbers. The existence of gravity waves with a rather long period, including solitary waves, was proved by Lavrent'ev [4]. Using the variational principle, Plotnikov [5] established that for an infinite set of Froude numbers, the problem has at least two different solutions.

In the present paper, we show that there is a range of Froude numbers (for waves of maximum amplitude) in which this problem has *three solutions*. The problem is solved by the method of complex boundary elements [6]. The accuracy of the method is checked by test calculations and comparison of the results obtained with results of other authors.

1. Formulation of the Problem. We study the problem of ideal, inviscid, incompressible fluid flow with free boundary C_1 along bottom C_3 consisting of rectilinear segments and a cylindrical bulge of radius R. The flow region D is bounded, in addition, by inlet C_2 and outlet C_4 (Fig. 1). This problem can be described by the Laplace equation

$$\Delta w(z) = 0, \qquad z = x + iy \in D$$

for the complex potential function $w(z) = \varphi(x,y) + i\psi(x,y)$, where $\varphi(x,y)$ is the velocity potential and $\psi(x,y)$ is the stream function, both satisfying the Cauchy–Riemann conditions. At the edges of the flow

Kemerovo State University, Kemerovo 650043. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 40, No. 1, pp. 27–35, January–February, 1999. Original article submitted May 26, 1997; revision submitted July 3, 1997.

 0021-8944/99/4001-0020 \$22.00

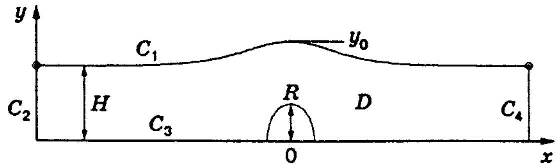

Fig. 1

region and at the bottom, the following boundary conditions are satisfied:

$$\operatorname{Im} w(z) = 0, \quad z \in C_3, \qquad \operatorname{Im} w(z) = \operatorname{Im} z, \quad z \in C_2, C_4.$$

The free boundary is a streamline ($\operatorname{Im} w = 1$), on which the Bernoulli relation

$$\left|\frac{dw}{dz}\right|^2 = 1 - 2(\operatorname{Im} z - 1)/\mathrm{Fr}^2, \quad z \in C_1, \tag{1.1}$$

is valid. Here $\mathrm{Fr} = V_\infty/\sqrt{gH}$ (H is the depth and V_∞ is the incoming flow velocity). The free surface C_1 is not known beforehand and should be found numerically in the course of solution of the problem.

2. Cauchy Integral Formula. It is known that for any analytic function w defined on a piecewise-smooth boundary $C = \bigcup\limits_{j=1}^{4} C_j$, the Cauchy integral formula

$$w(z_0) = \frac{1}{2\pi i}\int\limits_C \frac{w(z)}{z - z_0}\,dz, \quad z_0 \in D,$$

is valid.

In addition, the function $w(z)$ satisfies the Hölder–Lipschitz condition at the boundary C: $|w(z_1) - w(z_2)| < k|z_1 - z_2|^\alpha$, where $0 < \alpha \leqslant 1$, k is a constant, and z_1 and z_2 are any two points on the boundary C.

Then, it follows from the Sokhotskii formulas that for any point $z_0 \in C$, the relation

$$w(z_0) = w(z_0^+) - w(z_0^-) = \lim_{\xi \to z_0^+} \frac{1}{2\pi i}\int\limits_C \frac{w(z)}{z - \xi}\,dz = \frac{1}{2}w(z_0) + \text{v.p.}\frac{1}{2\pi i}\int\limits_C \frac{w(z)}{z - z_0}\,dz$$

is valid. Hence,

$$w(z_0) = \frac{1}{\pi i}\int\limits_C \frac{w(z)}{z - z_0}\,dz, \quad z_0 \in C.$$

The limit $\xi \to z_0^+$ implies that the point ξ tends to the point z_0, remaining inside the region D, $w(z_0^-) \equiv 0$.

Since, during the iterative process, the real part $\operatorname{Re} w(z)$ is known on the free surface and the imaginary part $\operatorname{Im} w(z)$ is known on the solid walls, for the function w, we have a mixed boundary-value problem. A numerical solution of this problem can be obtained by dividing the contour C into N linear elements Γ_j by nodes z_j ($j = \overline{1, N}$). Then, we have

$$w(z) = \lim_{\max|\Gamma_j| \to 0} G(z),$$

where $G(z)$ is a linear global test function for $z \in \sum\limits_{j=1}^{n} \Gamma_j$ and $G(z) = \sum\limits_{j=1}^{n} w_j \Lambda_j(z)$, where w_j is the value of $w(z)$ at the point z_j, and $\Lambda_j(z)$ is a linear basic function:

$$\Lambda_j(z) = \begin{cases} (z - z_j)/(z_j - z_{j-1}), & z \in \Gamma_{j-1}, \\ (z_{j+1} - z)/(z_{j+1} - z_j), & z \in \Gamma_j, \\ 0, & z \notin \Gamma_{j-1} \cup \Gamma_j. \end{cases}$$

After the indicated division and linear approximation of the function $w(z)$ on the boundary, the Cauchy

TABLE 1

$N(N_g)$	E_1	E_2	E_3	$\max\lvert y^{ex} - y^{n}\rvert$
72(30)	$5.5 \cdot 10^{-3}$	$2.0 \cdot 10^{-3}$	$1.6 \cdot 10^{-2}$	$1.3 \cdot 10^{-3}$
145(60)	$1.3 \cdot 10^{-3}$	$6.5 \cdot 10^{-4}$	$9.3 \cdot 10^{-3}$	$7.2 \cdot 10^{-4}$
290(120)	$3.1 \cdot 10^{-4}$	$6.2 \cdot 10^{-4}$	$8.7 \cdot 10^{-3}$	$5.4 \cdot 10^{-4}$
580(240)	$7.6 \cdot 10^{-5}$	$6.0 \cdot 10^{-4}$	$5.7 \cdot 10^{-3}$	$3.2 \cdot 10^{-4}$

integral can be calculated analytically in the sense of the principal value as $z \to z_j$. As a result, we obtain

$$2\pi i w_j = w_{j+1} - w_{j-1} + w_j \ln\left(\frac{z_{j+1} - z_j}{z_{j-1} - z_j}\right) + \sum_{\substack{m=1 \\ m \neq j, j+1}}^{N} I_m, \tag{2.1}$$

where

$$I_m = w_{m+1} - w_m + \left[\frac{(z_j - z_m)w_{m+1}}{z_{m+1} - z_m} - \frac{(z_j - z_{m+1})w_m}{z_{m+1} - z_m}\right] \ln\left(\frac{z_{m+1} - z_j}{z_m - z_j}\right).$$

Substituting the known real or imaginary parts of the function w at $j = \overline{1, N}$ into this equality, we obtain a system of N linear algebraic equations for N unknowns to determine $\operatorname{Re} w$ on C_3 and C_4 and $\operatorname{Im} w$ on C_1 and C_2. The accuracy of the calculation and the convergence of the method were verified by a procedure in [7]. It is required to find a solution of the Laplace equation in the region $D = \{0 \leqslant x \leqslant 2\pi;\ -1 \leqslant y \leqslant \sin(x)\}$, for which the nonpenetration condition $\operatorname{Im} w(z) = 0$ is imposed at the bottom and vertical walls, and the condition $\operatorname{Re} w(z) = -\cos(x)\cosh(y + 1)$, whose right side is a harmonic function, is imposed on the upper boundary. Table 1 lists the relative errors versus the number of nodes on the boundary (N is the number of nodes over the entire boundary and N_g is the number of nodes on the free boundary):

$$E_1 = \frac{\max\lvert\operatorname{Im} w^{ex} - \operatorname{Im} w^{n}\rvert}{\max\lvert\operatorname{Im} w^{ex}\rvert}, \qquad E_2 = \frac{\max\lvert V_x^{ex} - V_x^{n}\rvert}{\max\lvert V_x^{ex}\rvert}, \qquad E_3 = \frac{\max\lvert V_y^{ex} - V_y^{n}\rvert}{\max\lvert V_y^{ex}\rvert}.$$

Here w^{n}, V_x^{n}, and V_y^{n} are numerical values of the complex potential function and velocity vector components, and $\operatorname{Im} w^{ex}(x, y) = \sin(x)\sinh(y + 1)$, $V_x^{ex} = \sin(x)\cosh(y + 1)$, and $V_y^{ex} = -\cos(x)\sinh(y + 1)$ are exact values.

Double Nodes. The solution of hydrodynamic problems by numerical methods involves considerable difficulties in satisfying the boundary conditions at the angular points belonging simultaneously to the boundaries of the regions of definition of the real and imaginary parts of the function w. Incorrect handling of these nodal "singularities" affects the accuracy of the results and the stability of the algorithm in simulation of the free boundaries. In boundary-element methods, the indicated difficulties can be surmounted by introducing double nodes.

Let a double node be described by the identity $z_{m+1} \equiv z_m$. We assume that $z_m \in C_1$ and the potential $\operatorname{Re} w$ is defined at this node, and at the node $z_{m+1} \in C_2$, the function $\operatorname{Im} w$ is defined. By virtue of the continuity of the function w, at the double node the following natural conditions are satisfied:

$$\operatorname{Im} w_m = \operatorname{Im} w_{m+1}, \qquad \operatorname{Re} w_{m+1} = \operatorname{Re} w_m. \tag{2.2}$$

In this case, one should replace the mth and $(m + 1)$th lines in system (2.1) by condition (2.2). In addition, the elements of the mth and $(m + 1)$th columns of the matrix of system (2.1) do not contain contributions of integrals over the element Γ_m, which has zero length. At the points of intersection of the free boundary C_1 with the lateral segments C_2 and C_4 the boundary conditions change, and, hence, the problem is solved using the proposed procedure.

3. Algorithm of Constructing the Free Boundary. *Determination of the Potential.* Let the boundary C_1 be known in the kth approximation. To begin an iterative process, it is necessary to find the potential $\varphi(x, y) = \operatorname{Re} w(z)$ on the boundary C_1. From the Bernoulli relation (1.1), the velocity-vector magnitude is expressed as

$$\left|\frac{dw}{dz}\right| = q = \sqrt{1 + 2(1-y)/\mathrm{Fr}^2}. \tag{3.1}$$

Using the procedure proposed in [3], we introduce the parameter $V = v_0/v_\infty$. Then, Eq. (3.1) becomes

$$q = \sqrt{1 - (1 - V^2)\frac{y-1}{y_0 - 1}}, \tag{3.2}$$

where y_0 is the ordinate of the point on the free surface at which the velocity v_0 is given.

Since the boundary C_1 is a streamline, the velocity vector on it is tangent to the contour. Hence it follows that $q = \partial\varphi/\partial s$. Since the potential is determined with accuracy up to the additive constant, we set $\varphi_1 = 0$. Next, for any point on the free boundary, we have

$$\varphi_{i+1} = \varphi_i + \frac{q_i + q_{i+1}}{2}\Delta s_i, \tag{3.3}$$

where $i = \overline{1, N_g}$ are the nodal point numbers on the free boundary, $q_i = q(y_i)$ is given by formula (3.2), and $\Delta s_i = \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}$ is the length of the ith element of the free boundary.

Determination of the Free-Boundary Shape. The algorithm of determining the free boundary is as follows:

(1) let a certain position of the free boundary $C_1^{(k)}$ be known;

(2) the values of φ_i at nodes z_i on $C_1^{(k)}$ are determined from (3.3);

(3) the system of linear equations (2.1) is solved;

(4) the values of the velocity vector components are determined at points of the free boundary $C_1^{(k)}$

$$U_i = \mathrm{Re}\,\frac{dw}{dz}, \qquad V_i = -\mathrm{Im}\,\frac{dw}{dz};$$

(5) from the condition of collinearity of the velocity vector and the tangent to the boundary ($dy/dx = V/U$), the new position of the free boundary $C_1^{(k+1)}$ is calculated:

$$y_{i+1}^{k+1} = y_i^{k+1} + \Delta y_i^k.$$

Here the increment Δy_i^k is determined by expansion in the Taylor series:

$$\Delta y_i^k = \frac{V_i}{U_i}(x_{i+1} - x_i) + \frac{1}{2!}\frac{d}{dx}\left(\frac{V_i}{U_i}\right)(x_{i+1} - x_i)^2 + \ldots + \frac{1}{4!}\frac{d^3}{dx^3}\left(\frac{V_i}{U_i}\right)(x_{i+1} - x_i)^4.$$

The cycle is repeated until attainment of the required accuracy: $\max\limits_i |y_i^{k+1} - y_i^k| < \varepsilon$. Then, the Froude number is evaluated from the formula $\mathrm{Fr} = \sqrt{2(y_0 - 1)/(1 - V^2)}$. As a zero approximation, the straight line $y^0 = 1$ is used. The exception is the vicinity of the point y_0, in which the initial value $y_0^0 = 1 + 0.001$ is assigned.

The derivatives at points on the boundary of the region were calculated by schemes of high-order accuracy [8]. Testing the algorithm of constructing the free boundary by the procedure proposed in [3] showed the high accuracy and convergence rate. According to this procedure, the equation of the streamline $\psi = 1$ is derived from the velocity distribution on it by solving the problem of an ideal infinite fluid flow over a cylinder analytically. The deviations of the free boundary from the exact solution versus the number of points on the boundary, obtained as a result of five iterations, are given in Table 1.

The algorithm used to construct the free boundary was successfully employed previously [9]. The difference is, that in the present work, the increment Δy is found by expansion in a Taylor series. This reduces the number of iterations and increases the accuracy of the method, especially for limiting regimes.

4. Discussion of the Results. *Flow over Obstructions.* If the problem of fluid flow over obstructions is solved using the Bernoulli integral (3.1), it is possible to construct only a trivial solution that describes uniform flow with disappearance of the obstruction. In this case, the solution is valid for some values of

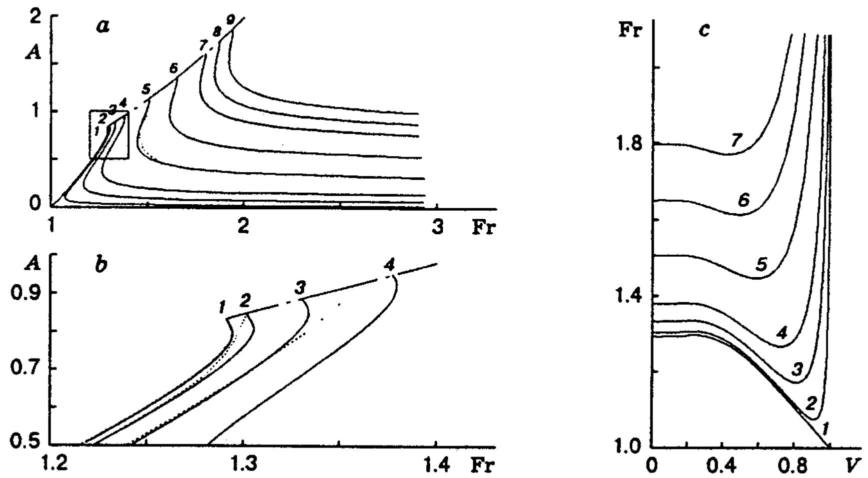

Fig. 2

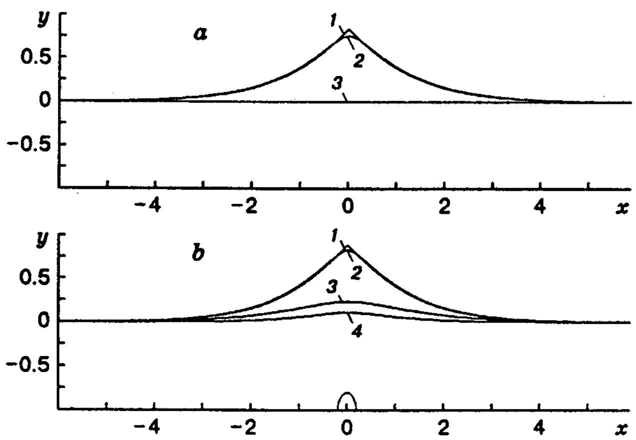

Fig. 3

the Froude number (Fr $\geqslant$ 1) that depend on the ratio R/H (R is the radius of the cylinder and H is the depth of the flow) and below these values a steady solution does not exist. This problem also admits a second solution, which was obtained in [3]. Vanden-Broeck [10] also sought two solutions and calculated the relation of between the wave amplitude and the Froude number for $R/H = 0.2$ and 0.5. He did not completely calculate the problem in the region of amplitudes close to the limiting value. Guzevskii [3] gives a detailed calculation only for $R/H = 0.1$ and does not note that the dependence of the amplitude $A = A(\mathrm{Fr})$ is multivalued for amplitudes close to the limiting values. Results of our calculations, which are given in Fig. 2, showed that the nonlinear problem of an ideal heavy fluid flow over an obstruction has an additional, *third* the solution in the region of limiting values of the wave amplitude. The nonuniqueness of the solution for a solitary wave is noted by Vanden-Broeck [10] and Longuet-Higgins and Fenton [11].

Figure 2a shows calculated values of the amplitude A versus the Froude number for different values of R/H. Curve 1 corresponds to $R/H = 0$ and describes a solitary wave, and curves 2–9 correspond to the ratio $R/H = 0.1$, 0.2, 0.3, 0.5, 0.7, 0.9, 1.0, and 1.1. The dot-and-dashed curve connects points of the curves that correspond to the maximum value $A_{\max} = \mathrm{Fr}^2/2$. The dotted curve near curve 5 corresponds to the calculations of [10]. The rectangle shows the region in which there is ambiguous behavior of the solution in the zone of the limiting wave. This region is scaled up in Fig. 2b, where the calculations for $R/H = 0$ (solitary waves) completely agree with the calculations of Maklakov [12] using his theory (the calculations for $R/H = 0.1$ are taken from [3], and the calculations for $R/H = 0.2$ are taken from [10]). The dependence

TABLE 2

A	Fr	K	P	T	M
0.10006	1.04663	0.02477	0.02391	0.04869	0.71044
0.19975	1.07004	0.04759	0.04522	0.09282	0.91130
0.24940	1.11523	0.10924	0.10063	0.20988	1.22732
0.30008	1.13691	0.14719	0.13371	0.28091	1.36221
0.34902	1.15716	0.18757	0.16823	0.35580	1.48121
0.39949	1.17735	0.23251	0.20596	0.43847	1.59362
0.44980	1.19674	0.27999	0.24512	0.52512	1.69567
0.49910	1.21616	0.33033	0.28533	0.61567	1.79163
0.54977	1.23407	0.38100	0.32554	0.70655	1.87257
0.59901	1.25017	0.42907	0.36317	0.79225	1.93812
0.64913	1.26556	0.47591	0.39891	0.87482	1.99159
0.70006	1.27938	0.51759	0.42978	0.94738	2.02689
0.73890	1.28779	0.54148	0.44693	0.98841	2.03693*
0.74988	1.29002	0.54685	0.45023	0.99709	2.03541
0.78046	1.29387	0.55413	0.45441	1.00854*	2.02286
0.79740	1.29457*	0.55232	0.45200	1.00433	2.00675
0.83328*	1.29095	0.53512	0.43784	0.97297	1.97147

Note. The asterisk distinguishes maximum values.

of the Froude number on the parameter V for the same values of R/H is given in Fig. 2c, which illustrates, although less vividly, the non-single-valued function $\mathrm{Fr} = \mathrm{Fr}(V)$ in the zone of limiting waves.

Figure 3a shows the free-surface shapes for $R/H = 0$ that correspond to three solutions for the same Froude number $\mathrm{Fr} = 1.2910$ at $A = 0.8332$, 0.7531, and 0 (curves 1–3), and Fig. 3b shows solutions for $R/H = 0.2$ and $\mathrm{Fr} = 1.3322$ at $A = 0.8875$, 0.8220, and 0.1106 (curves 1, 2, and 4). For the last case, the wave shape for the Froude number that is maximum for the first solution and the beginning of the second solution is also given, $\mathrm{Fr} = 1.1704$ and $A = 0.2355$ (curve 3).

Integral Characteristics of Solitary Waves. Determining integral characteristics such as the circulation C, mass M, and potential P and kinetic K energies is an important issue. These characteristics can be used to check the accuracy of the numerical method. Steady fluid flow was considered above. To characterize a solitary wave propagating on "calm" water, we introduce the new function $W(z) = \mathrm{Fr}(w(z) - z)$ ($\Phi = \mathrm{Re}\, W$ and $\Psi = \mathrm{Im}\, W$).

Longuet-Higgins and Fenton [11] proved that for solitary waves the following relations are valid:

$$K = \mathrm{Fr}(\mathrm{Fr}M - C)/2; \tag{4.1}$$

$$P = (\mathrm{Fr}^2 - 1)M/3. \tag{4.2}$$

These integral relations were used to check the accuracy of the solitary waves obtained since all quantities that enter in these equations were determined numerically from the formulas

$$P = \frac{1}{2}\int_a^b y^2\,dx = \frac{1}{6}\sum_{i=1}^{N_g}(y_i^2 + y_i y_{i+1} + y_{i+1}^2)(x_i - x_{i+1}); \tag{4.3}$$

$$K = -\frac{1}{2}\int_a^b \Phi\frac{\partial\Psi}{\partial s}\,ds = -\frac{1}{12}\sum_{i=1}^{N_g}(2\Phi_i p_i + \Phi_i p_{i+1} + \Phi_{i+1}p_i + 2\Phi_{i+1}p_{i+1})L_i; \tag{4.4}$$

$$M = \frac{1}{2}\Big(\int_a^b y\,dx - \int_a^b x\,dy\Big) = \frac{1}{4}\sum_{i=1}^{N_g}(y_{i+1} + y_i)(x_{i+1} - x_i) - (x_{i+1} + x_i)(y_{i+1} - y_i); \tag{4.5}$$

TABLE 3

Fr	A	M	P	K	Source
1.2861	0.8270	1.973	0.435	0.534	[15]
1.2881	0.8296	1.963	0.431	0.527	[16]
1.2909	0.8332	1.970	0.438	0.534	[17]
1.2909	0.8332	—	—	—	[10]
1.2909	0.8332	1.970	0.438	0.534	[14]
1.2909	0.8332	1.970	0.438	0.534	[13]
1.2910	0.8333	1.971	0.438	0.535	Present paper

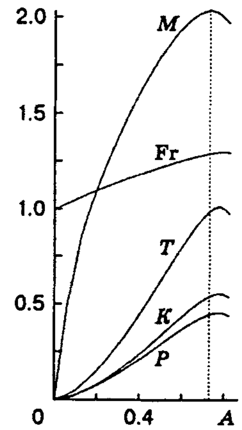

Fig. 4

$$C = \int_a^b \frac{\partial \Phi}{\partial s}\, ds = \Phi_b - \Phi_a. \tag{4.6}$$

Here a and b are the abscissas of the points of intersection of the free boundary with the boundaries of the regions C_2 and C_4, respectively, L_i is the length of the ith element, N_g is the number of points on the free surface, and $p_i = (\partial\Psi/\partial s)_i$.

The absolute error in calculating the potential and kinetic energies by formulas (4.1)–(4.5) is 10^{-3} for small and limiting waves and 10^{-5} for waves in the middle of the examined range. These estimates are obtained for a total number of elements of 290.

Table 2 gives the following calculated characteristics of solitary waves versus the amplitude: the Froude number, kinetic, potential, and total energies, and mass. These characteristics are shown in graphical form in Fig. 4. It is important to note that all maximum values of the solitary-wave characteristics are attained before the maximum amplitude is reached, and the maximum of the mass does not coincide with the maxima of the total energy and Froude number. This is also noted in [11].

Characteristic of Limiting Waves. The problem of constructing waves of limiting amplitude has been the subject of extensive research, but, by virtue of the complexity of this problem, many of the results obtained differ from each other. Some of them are shown in Table 3. Analysis of the cited papers leads to the conclusion that the most accurate calculations of solitary waves have been obtained by Sherykhalina [13] and Evans and Ford [14] ($A = 0.833199$ and Fr $= 1.290890$). The values of all characteristics obtained in our calculations differ from them by the fourth decimal place.

The first two lines of Table 3 gives the results obtained from the analytical relations for a solitary wave. Karabut [15] determined wave shapes on the basis of exact summation of Whiting series. Longuet-Higgins [16] obtained an approximate wave shape analytically using *a priori* characteristics of a solitary wave.

REFERENCES

1. O. M. Kiselev and L. M. Kotlyar, *Nonlinear Problems of the Theory of Jet Flows of a Heavy Fluid* [in Russian], Kazan' Univ., Kazan' (1978).
2. N. N. Moiseev, "Nonuniqueness of the possible shapes of steady heavy fluid flows at Froude numbers close to unity," *Prikl. Mat. Mekh.*, **21**, No. 6, 860–864 (1957).
3. L. G. Guzevskii, "Heavy fluid flow of finite depths over obstructions," in: *Dynamics of Continuous Media with Boundaries* (collected scientific papers) [in Russian], Izd. Chuvash. Univ., Cheboksary (1982), pp. 61–69.
4. M. A. Lavrent'ev, "Long-wave theory," *Prikl. Mekh. Tekh. Fiz.*, No. 5, 3–46 (1975).

5. P. I. Plotnikov, "Nonuniqueness of the solution of the problem of solitary waves and bifurcation of the critical points of smooth functionals," *Izv. Akad. Nauk SSSR, Ser. Mat.*, **55**, No. 2, 339–366 (1991).
6. T. V. Hromadka and Ch. Lai, *The Complex Variable Boundary Element Method in Engineering Analysis*, Springer-Verlag (1987).
7. A. G. Petrov and V. G. Smolyanin, "Calculation of unsteady waves on the surface of a heavy fluid of finite depth," *Prikl. Mat. Mekh.*, **57**, No. 4, 137–143 (1993).
8. K. E. Afanas'ev and T. I. Samoilova, "Procedure of using the boundary-element method in free-boundary problems," *Vychisl. Tekhnol.*, **4**, No. 11, 19–37 (1995).
9. A. G. Terent'ev and K. E. Afanas'ev, *Numerical Methods in Hydrodynamics: Handbook* [in Russian], Izd. Chuvash. Univ., Cheboksary (1987).
10. J. M. Vanden-Broeck, "Free-surface flow over an obstruction in a channel," *J. Phys. Fluids*, **30**, No. 8, 2315–2317 (1987).
11. M. S. Longuet-Higgins and J. D. Fenton, "On the mass, momentum, energy, and circulation of a solitary wave," *Proc. Roy. Soc. London*, **A340**, 471–493 (1974).
12. D. V. Maklakov, "Nonlinear theory of subcritical flows. Limiting flow regimes," Preprint No. 2, Kazan' Univ., Kazan' (1992).
13. N. M. Sherykhalina, "Development of Numerical Algorithms for Solving Hydrodynamic Problems with Singular Points on the Free Surface and Experimental Investigation of their Convergence Rate," Moscow (1995). Deposited at VINITI 12.16.95, No. 2550-B95.
14. W. A. B. Evans and M. J. Ford, "An exact integral equation for solitary waves (with new numerical results for some 'integral' properties)," *I. Proc. Roy. Soc. London*, **A452**, 373–390 (1996).
15. E. A. Karabut, "On the problem of a solitary wave on a fluid surface," *Dokl. Ross. Akad. Nauk*, **337**, No. 3, 339–341 (1994).
16. M. S. Longuet-Higgins, "On the mass, momentum, energy, and circulation of a solitary wave," *Proc. Roy. Soc. London*, **A337**, 1–13 (1974).
17. J. M. Williams, "Limiting gravity waves in water of finite depth," *Philos. Trans. R. Soc. London, Ser. A*, **303**, 139–188 (1981).